ENERGY ANALYSIS OF THE STABILITY OF PLANE-PARALLEL FLOWS WITH AN INFLECTION IN THE VELOCITY PROFILE

A. M. Sagalakov and V. N. Shtern

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Energy estimates are obtained for the critical Reynolds numbers for a number of flows having velocity profiles with an inflection point. Flows with a cubic velocity profile, a free submerged jet, and a jet in a channel are examined. It has been detected that the energy estimates are not more than two- to threefold less than the corresponding critical Reynolds numbers computed by linear theory.

1. There exist two approaches to the investigation of the stability of laminar stationary flows [1]. In linear stability theory the behavior of small perturbations is examined and conditions under which these perturbations grow with time are studied. The energy method permits the determination of under what conditions arbitrary perturbations will damp out monotonely. Both these approaches permit extraction of bands of parameters between the explicit stability and explicity instability domains within which the transition from the laminar to the turbulent flow modes or to another laminar flow mode is performed. This "theoretical band" turns out to be large for pressure flows in tubes and channels. Thus, for example, for Poiseuille flow in a plane channel, the energy analysis yields $R_*^*=49.9$ [2] and the linear theory $R_*^\circ=5772$ [3], while the experimental value of the critical Reynolds number is $R_*\sim1000$ (the Reynolds number is here determined over half the width of the channel and the maximum stream velocity). The so-called "paradoxical" effect of the viscosity is characteristic in these examples, i.e., the viscosity results here not only in energy dissipation but also contributed to the origination of growing Tollmien-Schlicting waves. The situation is different if the instability is due to the destabilizing influence of definite mass forces. For example, for a convective instability of the flow of a fluid heated from below [4], the critical parameters computed by an energy method turn out to be quantities of the same order as those given by linear theory. No difference between R_*^* and R_*° is generally obtained in a number of cases in [5] where the Taylor instability of the flow between rotating coaxial cylinders is considered.

This paper is devoted to an energy analysis of the stability of a number of plane-parallel flows with velocity profiles having an inflection point. An inviscid instability is characteristic here, and the viscosity turns out to be the ordinary stabilizing effect. The magnitudes of R_*° are comparatively low and, hence, it can be expected a priori that the energy estimates of the critical Reynolds numbers will not differ radically from the results of linear analysis. The values of R_*° were three- to fourfold greater than the corresponding values of R_*^* for a number of artificial profiles in [6].

An energy stability analysis first performed by Orr [7] and given a foundation by Serrin [8] consists of the fact that the equation for the energy of an arbitrary perturbation is considered. In the viscous fluid case it is

$$\frac{\partial E}{\partial t}\equiv\frac{\partial}{\partial t}\int\frac{v_iv_i}{2}\,d\Omega=-\int v_iv_j\frac{\partial U_i}{\partial x_j}\,d\Omega-\frac{1}{R}\int\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\Omega \qquad (1.1)$$

The variables are given here in dimensionless form, U_i and v_i are the components of the fundamental velocity field and the perturbation field, respectively. Integration is over some characteristic volume on

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whose surface the velocity perturbations vanish. It is furthermore assumed that v_i is not generally a solution of the equation for the perturbations, but is some arbitrary velocity field satisfying the mentioned boundary conditions and the continuity requirement

$$\partial v_i / \partial x_i = 0$$

Such a field is called a trial field. Serrin showed that if R is less than a definite quantity then $\partial E/\partial t < 0$ for any trial velocity field. Under these conditions the energy of any perturbation will decrease monotonely with time. The purpose of the energy analysis is to determine the least Reynolds number $R_*{}^*$ for which $\partial E/\partial t$ will first vanish for some trial field (or some trial fields). In other words, find the R for which

$$\max_{v_i} \partial E / \partial t = 0$$

For $R = R_*{}^*$ the energy of an arbitrary perturbation will diminish strictly monotonely, with the exception of some times when $\partial E/\partial t = 0$. The Euler-Lagrange equations for this variational problem are

$$\Delta v_i = \frac{R}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) v_j + \frac{\partial \varphi}{\partial x_i}, \qquad \frac{\partial v_i}{\partial x_i} = 0 \tag{1.2}$$

Here φ is a Lagrange multiplier which is the analog of the hydrodynamic pressure.

2. Turning to the case of a plane channel and performing a Fourier transformation in the coordinates x and z (the direction of the x axis agrees with the stream direction), we obtain an eigenvalue problem in R

$$\begin{gathered} u'' - k^2 u = {}^1/_2 RU'v + i\alpha\lambda, \quad v'' - k^2 v = {}^1/_2 RU'u + \lambda' \\ w'' - k^2 w = i\beta\lambda \quad (-1 \leqslant y \leqslant 1) \\ i\alpha u + i\beta w + v' = 0 \quad (u(\pm 1) = v(\pm 1) = w(\pm 1) = 0) \end{gathered} \tag{2.1}$$

Here u, v, w, λ are (x, y, z) components of the Fourier transforms of v_i and φ; α, β, k are the x and z components of the wave vector and its modulus, respectively, and the prime denotes differentiation with respect to y. It is convenient to reduce the system (2.1) to one equation. Introducing the angle θ between the main flow velocity vector and the perturbation wave vector (hence $\alpha = k \cos\theta$, $\beta = k \sin\theta$), we obtain

$$\begin{gathered} \left(\frac{d^2}{dy^2} - k^2\right)\left[\frac{1}{U'}\left(\frac{d^2}{dy^2} - k^2\right)^2 v + ikR\cos\theta\left(v' + \frac{U''}{2U'}v\right)\right] + \frac{k^2R^2\sin^2\theta}{4} U'v = 0 \\ v = v' = v'''' - 2k^2 v'' = 0 \quad \text{at} \quad y = \pm 1 \end{gathered} \tag{2.2}$$

Let us list some useful properties of the problem.

1. There exist only real eigenvalues of R. In order to see this, it is sufficient to multiply the first three equations in (2.1) by u*, v*, w*, respectively (here the asterisk denotes the complex conjugate), to add them, and integrate with respect to y over the whole interval.

2. In addition to the eigenvalue R and the eigenfunction v there also exists the eigenvalue −R with the eigenfunction v*. In order to see this, it is sufficient to take the complex conjugate in (2.2). It therefore is meaningful to speak only of seeking the minimum of the absolute value of the eigenvalue R.

3. It is sufficient to study the behavior of R in the quadrant $0 \le \theta \le {}^1/_2\pi$ ($\alpha > 0$, $\beta > 0$), since it is seen from (2.2) that a change in sign of θ as well as a π change in θ with a subsequent complex conjugate of (2.2) will result in the very same spectral problem.

4. For $k \ll 1$ the eigenvalues are $R \sim 1/k$. This can be seen by introducing $R_+ = kR$ and then neglecting terms containing k in (2.2).

5. The case $k \gg 1$ merits more detailed discussion. Making the transformation $R = kR^+$, $y_+ = ky$, and then allowing k to tend to infinity, we obtain from (2.2)

$$\begin{gathered} \left(\frac{d^2}{dy_+^2} - 1\right)\left[\frac{1}{U_+'}\left(\frac{d^2}{dy_+^2} - 1\right)^2 v_+ + iR^+\cos\theta\left(v_+' + \frac{U_+''}{2U_+'}v_+\right)\right] + \frac{R^{+2}\sin^2\theta}{4} U_+'v_+ = 0 \\ v_+ = v_+' = v_+''' - 2v_+'' = 0 \quad \text{at} \quad y_+ = \pm\infty \\ v_+ = v(y_+), \qquad U_+ = U(y_+) \end{gathered} \tag{2.3}$$

If the eigenvalue problem (2.3) has non-trivial solutions, then $R \sim k$ asymptotically for $k \gg 1$. However, in some cases (2.3) has only a trivial solution and the absolute value of R increases more rapidly with the growth of k than according to a linear law.

The last two properties indicate that R reaches a minimum for $k \sim 1$, i.e., when the wave number is on the order of the reciprocal characteristic dimension of the problem (moderate values of U' are in mind).

3. The eigenvalues are computed numerically in the domain $k \sim 1$ until R becomes an asymptotic dependence for small and large k. For large R a small parameter appears in the highest derivative in (2.2), which results in the difficulties customary in such cases. Although the energy estimates $R_*{}^*$ are comparatively small, as a rule, it should be kept in mind that there is an R^2 term in (2.2). For a detailed spectral analysis it is generally desirable to have a universal numerical method suitable for both small as well as large R. The differential factorization method satisfies these conditions [9]. Performing a great number of eigenvalue computations on an electronic computer demands the selection of an economical factorization scheme. The standard modification [9] results for (2.2) in very awkward right sides in the system of equations for the factorization coefficients. M. A. Gol'dshtik and V. A. Sapozhnikov proposed the following approach to a problem of similar nature. The factorization method is not related to the kind of boundary conditions, but is found from the demand for greatest simplicity in the system of differential equations for the factorization coefficients. The initial data for this system are obtained by using integration of the initial linear equation in a small interval of boundary points. These assumptions are realized as follows in application to (2.2).

Let us introduce the function

$$\psi = \frac{2}{RU'}\left(\frac{d^2}{dy^2} - k^2\right)^2 v - 2i\alpha\left(v' - \frac{U''}{2U'}\,v\right) \tag{3.1}$$

By virtue of the boundary conditions (2.2) $\psi(\pm 1) = 0$. Let us define the factorization scheme by the relationship

$$\begin{pmatrix} v''' \\ \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} v \\ v' \\ v'' \end{pmatrix} \tag{3.2}$$

We obtain the system of differential equations for A_{ij} by differentiating (3.1) and using (2.2) and (3.1). It is

$$\begin{gathered} A_{11}' = {}^1/_2 RU'A_{21} - A_{11}A_{13} + {}^1/_2 i\alpha RU'' - k^2 \\ A_{12}' = {}^1/_2 RU'A_{22} - A_{11} - A_{12}A_{13} - i\alpha RU' \\ A_{13}' = {}^1/_2 RU'A_{23} - A_{12} - A_{13}^2 + 2k^2, \\ A_{21}' = A_{31} - A_{11}A_{33}, \quad A_{22}' = A_{32} - A_{21} - A_{12}A_{23} \\ A_{23}' = A_{33} - A_{22} - A_{13}A_{23} \\ A_{31}' = k^2 A_{21} - A_{11}A_{33} - {}^1/_2 \beta^2 RU' \\ A_{32}' = k^2 A_{22} - A_{31} - A_{12}A_{33}, \quad A_{33}' = k^2 A_{23} - A_{32} - A_{13}A_{33} \end{gathered} \tag{3.3}$$

The Cauchy problem is considered for the system (3.3). The initial conditions near the boundary points are determined by using (2.5) and (2.2). Denoting the results of integrating (3.3) from one boundary point by the subscript plus below, and from the other by the subscript minus, we obtain the characteristic equation for R:

$$F(R) \equiv \det(A_+ - A_-) = 0$$

at any inner point of the interval (−1,1). In a number of cases it is useful to select the root of the equation $U'(y) = 0$ as such a point if it lies within the interval, since this point is the analog of the critical point in the linear theory of hydrodynamic stability. For symmetric U profiles it is sufficient to consider symmetric and antisymmetric solutions of (2.2). In this case the characteristic equation is

$$F(R) \equiv A_{32}(A_{11}A_{23} - A_{13}A_{21}) = 0$$

on the channel axis $y = 0$. For the antisymmetric solutions $A_{32} = 0$, and the expression in the parentheses vanishes in the symmetric case.

For some values of k and θ the least positive root of $F(R) = 0$ is found initially, and then a sufficiently compact mesh $R(k, \theta)$ is constructed by continuity, and the magnitude and position of the minimum of this function are determined. If need be, these parameters are refined. The numerical analysis did not disclose any intersection of the spectral branches for the considered flows.

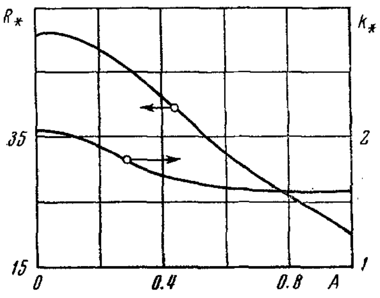

Fig. 1

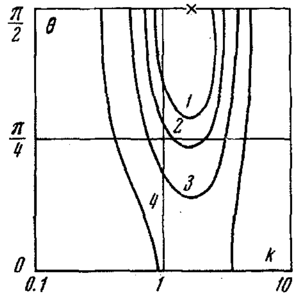

Fig. 2

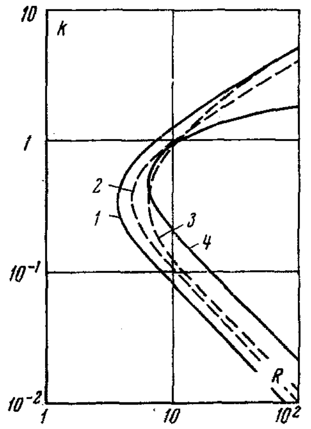

Fig. 3

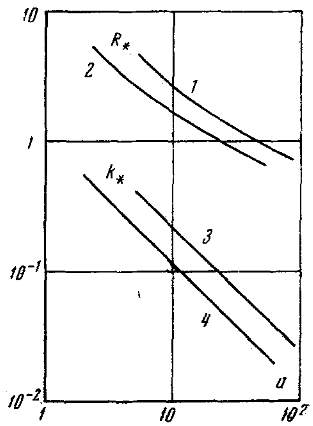

Fig. 4

The computations were carried out on a BÉSM-6. The eigenvalues were found to prescribed accuracy (three significant figures).

A computation of Couette-Poiseuille flow stability

$$U = (1 - A)(1 - y^2) + Ay$$

was mainly used to check the numerical method used. In the Poiseuille flow case $(A = 0)$ the calculations yielded $R_*^* = 49.9$, $k_* = 2.04$, which agrees with the results obtained earlier [2], while $R_*^* = 20.6$, $k_* = 1.56$ was found for Couette flow $(A = 1)$, as in [4]. The dependences $R_*(A)$ (curve 1) and $k_*(A)$ (curve 2) are represented in Fig. 1. The energy estimates of R_* diminish as A grows, in contrast to the linear theory results [10] (for small A an increase in R_* still holds; however, it is quite insignificant). In all the cases considered here and below, the minimum R is achieved for $\theta = \pi/2$, i.e., the perturbations whose wave vector is perpendicular to the main flow velocity vector are most "dangerous." No analytic proof of this fact has been obtained, however.

4. The flow of a viscous, heat-conducting fluid between vertical parallel planes heated to different temperatures is an example of a plane-parallel nonsymmetric flow with an inflection in the velocity profile. The velocity profile has the form

$$U = y - y^3$$

If the Prandtl numbers are small, it is possible to limit oneself to purely hydrodynamic perturbations in the stability analysis. Within the scope of linear theory this analysis has been carried out in [11], and later in [12]. The critical parameters are $R_*^\circ = 82$, $\alpha_* = k_* = 1.32$.

The energy analysis yields $R_*^* = 29$, $\beta_* = k_* = 1.8$. If only two-dimensional perturbations are examined, then $R_*^* = 58$, $\alpha_* = 1.9$. The level lines $R(k, \theta) = \text{const}$ are presented in Fig. 2. Here the asterisk denotes the position of minimum R. The level lines (curves 1, 2, 3, 4) correspond to $R = 35, 40, 50, 80$. The level line picture is typical for all the flows considered herein.

The critical numbers R_*° and $R_*{}^*$ differ by less than threefold, while the difference for pressure flow is several orders of magnitude. The experimental measurements [12] which were conducted for the Prandtl number $P = 0.71$ yielded $R_* = 91\ (1 \pm 0.1)$, $k_* = 1.37$, which somewhat exceeds the linear theory results. This can be associated with errors in the tests.

Still smaller discrepancies between $R_*{}^*$ and R_*° are obtained for jet type flows.

Let us examine the stability of the flow

$$U = 1 - \mathrm{th}^2 y \qquad (-a \leqslant y \leqslant a) \tag{4.1}$$

The velocity profile (4.1) is a self-similar Schlichting solution for a plane submerged jet. As before, the adhesion conditions at $y = \pm a$ are the boundary conditions for the perturbations here. This flow, which we provisionally call a jet in a channel, is considered here as the model of a symmetric flow with an inflection point in the velocity profile. As is seen from (4.1), the Reynolds number is defined by means of the jet halfwidth and the maximum velocity.

A number of results for the case $a = 6$ is presented in Fig. 3. Curve 1 is the dependence R(k) computed by an energy method for $\theta = \frac{1}{2}\pi$. Curve 2 corresponds to plane perturbations ($\theta = 0$). The dependence R(k) for $\theta = \frac{1}{2}\pi$ of the next spectral branch (antisymmetric perturbations in v) is pictured by the curve 3. A neutral curve (curve 4) computed by linear stability theory is presented in Fig. 3 for comparison. It corresponds to perturbations with $\theta = 0$; by virtue of the Squire theorem [1], they are most dangerous in the linear case. Let us present values of the critical parameters for comparison. The energy method ($\theta = \frac{1}{2}\pi$, absolute minimum) yields $R_*{}^* = 3.7$, $k_* = 0.327$, the energy method ($\theta = 0$), $R_* = 4.75$, $k_* = 0.345$, and linear theory ($\theta = 0$), $R_* = 6.25$, $k_* = 0.46$.

The asymptotic behavior of R(k), determined by the energy method, has the general form for all the spectral branches considered and for all θ. In conformity with property 4 (see Sec. 2), R is inversely proportional to k for small k, while R increases more rapidly for large k than according to the linear law. Here R is approximately proportional to k^2. The minimum value of R for antisymmetric perturbations, or 6.1, is reached for $k = 0.4$, i.e., for a rather larger wave number as compared with the case of symmetric perturbations. For large k the difference between the Reynolds numbers for symmetric and antisymmetric perturbations tends to zero. This indicates that the shortwave perturbations differ from zero in practice only in some subdomain of the interval $-1 \leq y \leq 1$, and this subdomain does not include the point $y = 0$, so that the symmetry or antisymmetry conditions are inessential on the axis.

Computations showed that the energy estimates of the critical Reynolds number depend substantially on the ratio between the channel width and the jet width a, and the Reynolds number tends to zero as $a \to \infty$. Presented in Fig. 4 is the dependence $R_*(a)$ (curve 1) in the band $5 \leq a \leq 100$. For $a > 10$, the relationship $R_* \sim a^{-0.55}$ is satisfied approximately. It is characteristic that the dependence of the critical wave number on a curve (curve 3) has the form $k_* \sim 1/a$, i.e., the wave number computed over to the halfwidth of the channel remains constant and approximately equal to 2.43 as the jet thins out, while the Reynolds number aR computed over the channel halfwidth grows as the jet thins. For sufficiently large wave numbers, the eigenvalue R is already independent of the channel width. Stratification of the curves R(k) with respect to a is not observed in the considered band for $k > 1$. This indicates that the subdomain where the shortwave perturbations are different from zero in practice does not include the boundary points but is apparently concentrated in the neighborhood of the inflection point of the velocity profile. But the critical numbers R_* correspond to too small k in this case and depend on the channel width.

Since the critical perturbations "sense" the channel walls, i.e., for them it is essential at what distance the boundary conditions are formulated, then the nature of the boundary conditions should visibly also affect the results.

It is interesting to examine a jet in unbounded space, and even more so since a number of linear stability theory papers has been devoted to it. The self-similar Schlichting solution for a jet

$$U(y) = \begin{cases} 1 - \mathrm{th}^2 a & (y \leqslant -a) \\ 1 - \mathrm{th}^2 y & (-a \leqslant y \leqslant a) \\ 1 - \mathrm{th}^2 a & (y \geqslant a) \end{cases}$$

is also taken as the basis here.

The boundary conditions are posed at infinity and reduce to the demand for damping. They can be referred to $y=\pm a$. For $|y|>a$ $U'=0$ in the system (2.1), by eliminating λ, it is possible to arrive at

$$v'''' - 2k^2 v'' + k^4 v = 0, \qquad u'' - k^2 u = \frac{i\alpha}{k^2}(v''' - k^2 v') \tag{4.2}$$

It is easy to see that three linearly independent solutions of (4.2) are found which damp out at infinity. For example, we have for $y<-a$

$$v = e^{ky}, \quad u = 0 \tag{4.3}$$

$$v = ye^{ky}, \quad u = \frac{i\alpha}{k} ye^{ky} \tag{4.4}$$

$$v = 0, \quad u = e^{ky} \tag{4.5}$$

These fundamental solutions can be used to calculate the factorization coefficients A_{ij}, by continuing them within the interval in conformity with (2.2) and by solving (3.2) at the point $y=-a+\varepsilon$, where ε is some small quantity (see Sec. 2).

Presented in Fig. 4 are the dependences $R_*(a)$ (curve 2) and $k_*(a)$ (curve 4) in the case of a free jet. The critical Reynolds numbers are approximately one and one-half times less than for a jet in a channel. As before the quantities k_* decrease proportionately to a, where $k_* a = 1.24$. It is characteristic that the relative difference between the results for R_* and k for a free jet and a jet in a channel does not tend to zero as $a \to \infty$.

The linear analysis also discloses a dependence of R_*° on a.

Thus, $R_*^\circ = 7.5$ is obtained in [13] for $a=3$, and $R_*^\circ = 4$ in [14] for $a=6$. The energy method yields $R_*{*} =$ 3.9 for $a=3$, and $R_*{*} = 2.2$ for $a=6$ in these cases.

On the basis of the analysis conducted, it can be concluded that, for a number of velocity profiles having an inflection point, the critical Reynolds numbers computed by the energy method are less than the corresponding quantities R_*° by not more than two- to threefold. Therefore, in these cases the energy method in combination with linear theory permits obtaining estimates of the critical Reynolds numbers which are satisfactory for a number of engineering applications.

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LITERATURE CITED

1. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Pt. 1, Nauka, Moscow (1965).
2. D. D. Joseph and S. Carmi, "Stability of Poiseuille flow in pipes, annuli, and channels," Quart. Appl. Math., 26, No. 4 (1969).
3. V. A. Sapozhnikov and V. N. Shtern, "Numerical analysis of the stability of plane Poiseuille flow," Zh. Prikl. Mekh. i Tekh. Fiz., No. 4 (1969).
4. D. D. Joseph, "Nonlinear stability of the Boussinesq equations by the method of energy," Arch. Ration. Mech. and Analysis, 22, No. 3, 163-184 (1966).
5. D. D. Joseph and B. R. Munson, "Global stability of spiral flow," J. Fluid Mech., 43, Pt. 3 (1970).
6. S. Carmi, "Energy stability of channel flows," ZAMP, 20, No. 4 (1969).
7. W. McF. Orr, "The stability or instability of the steady motions of a liquid," Proc. Roy. Irish Acad., Ser. A, 27, 69-138 (1907).
8. J. Serrin, "On the stability of viscous fluid motions," Arch. Ration. Mech. and Analysis, 3, No. 1 (1959).
9. V. A. Sapozhnikov, "Solution of the eigenvalue problem for ordinary differential equations by the factorization method," Transactions of the All-Union Seminar on Numerical Methods of Viscous Fluid Mechanics, Kanev, 1968, Vol. 2, Nauka, Novosibirsk (1969).
10. M. C. Potter, "Stability of plane Couette-Poiseuille flow," J. Fluid Mech., 24, Pt. 3 (1966).
11. R. V. Birikh, "On small perturbations of plane-parallel flow with cubic velocity profile," Prikl. Mat. i Mekhan., 30, No. 2 (1966).
12. C. M. Vest and V. S. Aspaci, "Stability of natural convection in a vertical slot," J. Fluid Mech., 36, Pt. 1 (1969).
13. I. P. Suprunenko, "Stability of jet flows," Izv. Akad. Nauk SSSR, Mekhanika, No. 4 (1965).
14. Ko Sung Hwan and M. Lessen, "Viscous instability of an incompressible full jet," Phys. Fluids, 12, No. 11 (1969).